Indian Statistical Institute
Semestral Examination
Algebra II - BMath I

Max Marks: 60 Time: 180 minutes.

Answer all questions. You may use Theorems stated/proved in the class after correctly stating them. You may use results not discussed in the class only after proving them.

(1) (a) Show that there do not exist linear maps $T, S : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that $(TS - ST)(v) = v$ for all $v \in \mathbb{R}^n$. [5]
(b) State and prove the Rank-Nullity theorem. [6]
(c) Let V denote the vector space of all $n \times n$ real matrices and W the subspace of matrices with trace zero. Find a subspace W' such that $V = W \oplus W'$. [9]

(2) (a) Define the notions of : eigenvector, eigenvalue of an operator. Prove or disprove the existence of the following:
(i) A linear operator $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ having no eigenvalues.
(ii) A linear operator $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ with $1, 2$ as the eigenvalues. If such an operator exists, is it always invertible?
(iii) A linear operator $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ that is not diagonalizable. [6]
(b) Let A be a $n \times n$ orthogonal matrix with n odd. Show that either $+1$ or -1 is an eigenvalue of A. [6]
(c) Prove that a linear operator $T : V \longrightarrow V$ is nilpotent if and only if there is a basis of V such the matrix of T is upper triangular with diagonal entries zero. [8]

(3) (a) Decide whether the the matrix

$$A = \begin{pmatrix} 3 & 2-i & -3i \\ 2+i & 0 & 1-i \\ 3i & 1+i & 0 \end{pmatrix}$$

is diagonalizable. If it is, then find a unitary matrix P such that PAP^* is diagonal. [6]
(b) Define the following notion : positive definite symmetric bilinear form. Let V denote the vector space of all $n \times n$ real matrices. Show that for $A, B \in V$,

$$\langle A, B \rangle = \text{trace}(A^t B)$$

defines a positive definite symmetric bilinear form on V. Find a orthonormal basis of V with respect to the above form. [6]
(c) Recall that a $n \times n$ real matrix is said to be positive definite if the associated bilinear form $X^t AY$ on $\mathbb{R}^n$ is positive definite. Show that a real symmetric $n \times n$ matrix A is positive definite if and only if all its eigenvalues are greater than zero. [8]